



NCC-003-1162002 Seat No. _____

M. Sc. (Sem. II) (CBCS) Examination

April / May - 2017

Mathematics : CMT-2002

(Complex Analysis)

Faculty Code : 003

Subject Code : 1162002

Time : $2\frac{1}{2}$ Hours]

[Total Marks : 70

Instructions : (1) Answer all questions.

(2) Each question carries 14 marks.

1 Answer any **seven** questions :

2×7=14

(i) Define the stereographic projection $T^{-1} = S \rightarrow \mathbb{C}_\infty$.
What is its inverse ?

(ii) True or false ? Justify. $f: \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = \bar{z}$
is differentiable.

(iii) If $s(z) = \frac{az+b}{cz+d}$ is a bilinear transformation then find
 s^{-1} .

(iv) How many fixed points can a bilinear transformation
 $S \neq L$ have ? Justify.

(v) If $\gamma: [0,1] \rightarrow \mathbb{C}$ is defined by $\gamma(t) = a + re^{i2\pi nt}$, $\forall t \in [0,1]$,
for some $a \in \mathbb{C}, r > 0$ and $n \in \mathbb{Z}$ then find $V(\gamma)$.

(vi) If G is a region in \mathbb{C} , $f: G \rightarrow \mathbb{C}$ is analytic and $f(G) \subset \Gamma$
for some circle Γ in \mathbb{C} the prove that $f: G \rightarrow \mathbb{C}$ is a
constant function.

(vii) True or false ? Justify. Every bilinear transformation
 S can be uniquely written as $Sz = \frac{az+b}{cz+d}$, for some
 $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.

(viii) If $\log z = \log r + i\theta$, $r > 0$, $\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$ then prove that

$$\log i^2 \neq 2 \log i.$$

(ix) If $f: \mathbb{C} \rightarrow \mathbb{C}$ is an entire function and

$$f\left(\frac{1}{n}\right) = 0, \forall n = 1, 2, \dots \text{ then prove that } f \equiv 0 \text{ on } \mathbb{C}.$$

(x) If $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $f: D \rightarrow D$ is analytic, $f(0) = 0$,

$$f\left(\frac{1}{2}\right) = \frac{1}{2} \text{ then prove that } f(z) = z, \forall z \in D.$$

2 Answer any **two** questions :

2×7=14

- (a) If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct then for $z_1 \in \mathbb{C}_\infty$, define the cross ratio (z_1, z_2, z_3, z_4) . Prove that every bilinear transformation preserves cross ratio.
- (b) State and prove orientation principle.
- (c) Find the bilinear transformation taking $1 \rightarrow i, 0 \rightarrow 1, \infty \rightarrow 0$.

3 (a) Define rectifiable path in \mathbb{C} . Give an example of a path which is not rectifiable. **7**

(b) Define the winding number $n(\gamma, a)$ of a closed rectifiable path γ w.r.t. $a \notin \{\gamma\}$ and give the geometric meaning of the winding number. **7**

OR

(c) State, without proof, Cauchy's theorem for an open disc and find $\int_{\sigma} \frac{dz}{z^2 - 1}$, where $\sigma(t) = 1 + e^{it}, \forall t \in [0, 2\pi]$. **7**

(d) State and prove Morera's theorem. **7**

4 Answer any two questions :

2×7=14

(a) If $G \subset \mathbb{C}$ is a region $f: G \rightarrow \mathbb{C}$ is analytic,

a_1, a_2, \dots, a_n are the only zeros of f in G repeated according to the multiplicities, γ is a closed rectifiable curve in G , γ does not pass through

" a_k " $k=1, 2, \dots, m$ and $n(\gamma; w) = 0, \forall w \in \mathbb{C} \setminus G$ then

prove that $\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma; a_k)$. Deduce that

$$\int_{|z|=2} \frac{2z+1}{z^2+z+1} dz = 4\pi i.$$

(b) If $f: B(a, R) \rightarrow \mathbb{C}$ is analytic and $f(a) = 0$ then prove that " a " is a zero of multiplicity $n \in \mathbb{N}$ iff

$f^{m-1}(a) = f^{m-2}(a) = \dots = f'(a) = f(a) = 0$ and $f^n(a) \neq 0$.

(c) Show that for $a > 1, \int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}$.

5 Answer any two questions :

2×7=14

(a) Define essential singularity of a complex function of a complex variable and give an example. Prove that $a \in \mathbb{C}$ is an essential singularity of f iff $a_n \neq 0$, for infinitely many negative integers n in the Laurent's

expansion $\sum_{n=-\infty}^{\infty} a_n (z-a_n)^n$ of f at " a ".

(b) Find the Laurent's expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ in

(i) $ann(0; 1, 2)$ (ii) $ann(0; 2, \infty)$

(c) State, without proof Rouché's theorem and deduce the fundamental theorem of algebra from it.

(d) State and prove Argument principle.
